How Hard is Computing Parity with Noisy Communications?*

Chinmoy Dutta[†] Yashodhan Kanoria[‡] D. Manjunath[§] Jaikumar Radhakrishnan[¶]

Abstract

We show a tight lower bound of $\Omega(N\log\log N)$ on the number of transmissions required to compute the parity of N input bits with constant error in a noisy communication network of N randomly placed sensors, each having one input bit and communicating with others using local transmissions with power near the connectivity threshold. This result settles the lower bound question left open by Ying, Srikant and Dullerud (WiOpt 06), who showed how the sum of all the N bits can be computed using $O(N\log\log N)$ transmissions. The same lower bound has been shown to hold for a host of other functions including majority by Dutta and Radhakrishnan (FOCS 2008).

Most works on lower bounds for communication networks considered mostly the full broadcast model without using the fact that the communication in real networks is local, determined by the power of the transmitters. In fact, in full broadcast networks computing parity needs $\theta(N)$ transmissions. To obtain our lower bound we employ techniques developed by Goyal, Kindler and Saks (FOCS 05), who showed lower bounds in the full broadcast model by reducing the problem to a model of noisy decision trees. However, in order to capture the limited range of transmissions in real sensor networks, we adapt their definition of noisy decision trees and allow each node of the tree access to only a limited part of the input. Our lower bound is obtained by exploiting special properties of parity computations in such noisy decision trees.

1 Introduction

Since inexpensive wireless technology and sensing hardware have become widely available and are heavily used, much recent effort has been devoted to developing models for these networks and protocols based on these models. A wireless sensor network consists of sensors that collect and cooperatively process data in order to compute some global function. The sensors interact with each other by transmitting wireless messages based on some protocol. The protocol is required to tolerate errors in transmissions since wireless messages typically are noisy.

In the problem we study, each sensor is required to detect a bit; then, all the sensors are required to collectively compute the parity of these bits. The difficulty of this task, of course, depends on the noise and the connectivity of the network. In this paper, we assume that each bit sent is flipped (independently for each receiver) with probability $\epsilon > 0$ during transmission. As for connectivity, we adopt the widely used model of random planar networks. Here the sensors are placed randomly and uniformly in a unit square. Then each transmission is assumed to be received (with noise) by the sensors that are within some prescribed radius of the sender. The radius is determined by the amount of power used by the sensors, and naturally one wishes to keep the power used as low as possible, perhaps just enough to ensure that the entire network

^{*}A preliminary version of this work appeared in the Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 2008, pp. 1056-1065.

[†]Twitter Inc., San Francisco, USA. email: chinmoy@twitter.com

[‡]Columbia Business School, New York, USA. email: ykanoria@columbia.edu The work was done while this author was at Indian Institute of Technology, Mumbai, INDIA.

[§]Indian Institute of Technology, Mumbai, INDIA. email: dmanju@ee.iitb.ac.in

Tata Institute of Fundamental Research, Mumbai, INDIA. email: jaikumar@tifr.res.in

is connected. If the network is not connected then it cannot be expected to compute a function like parity which depends on all the input bits. It has been shown by Gupta and Kumar [7] that the threshold radius for connectivity is $\theta\left(\sqrt{\frac{\ln N}{N}}\right)$ for a random planar network of N sensors placed in a unit square. With a radius much smaller than this the network will not be connected almost surely, and with radius much larger it will be connected almost surely.

Our work is motivated by a protocol presented by Ying, Srikant and Dullerud [12] for computing the sum of all the bits (and hence any symmetric functions of these bits). They showed that even with radius of transmission just near the connectivity threshold, and constant noise probability, one can compute the sum using a total of $O(N \log \log N)$ transmissions. They observed the (trivial) lower bound of N transmissions (for every sensor must send at least one message), but left open the possibility of better upper bounds. One can compute the parity of the input bits from their sum; in fact, Ying et al. suggested that parity computation might be significantly easier than computing the sum. In this work, we prove a lower bound showing that the protocol of Ying et al. is optimal up to constant factors for computing the parity (and hence, also the sum) of the input bits. In order to state our result formally we need to define the model of noisy communication networks.

Definition 1 (Noisy communication network and protocol). A communication network is an undirected graph G whose vertices correspond to sensors and edges correspond to communication links. A message sent by a sensor is received by all its neighbors.

Noise: In an ϵ -noise network, the messages are subjected to noise as follows. Suppose sensor v sends bit b in time step t. Each neighbor of v then receives an independent noisy version of v; that is, the neighbor v of v receives the bit v0 v1, where v2, where v3 v4 is an v5-noisy bit (that takes the value v4 with probability v5 and v6 with probability v7 v7. These noisy bits being mutually independent for different neighbors.

Input: An input to the network is an assignment of bits to the sensors, and is formally an element of $\{0,1\}^{V(G)}$.

Protocol: A protocol on G for computing a function $f:\{0,1\}^{V(G)} \to \{0,1\}$ works as follows. The sensors take turns to send single bit messages, which are received only by the neighbors of the sender. In the end, a designated sensor $v^* \in V(G)$ declares the answer. The cost of the protocol is the total number of bits transmitted. A message sent by a sensor in some time step is a function of the bits that it possesses, which include its input bit and the noisy copy of the bits transmitted by its neighbors until then. The protocol with cost T is thus specified by a sequence of T vertices $\langle v_1, v_2, \ldots, v_T \rangle$ and a sequence of T functions $\langle g_1, g_2, \ldots, g_T \rangle$, where $g_t: \{0,1\}^{j_t} \to \{0,1\}$ and j_t is the number of bits possessed by v_t before time step t. Furthermore, $v_T = v^*$, and the final answer is obtained by computing g_T . Note that in our model the number of transmissions is the same for all inputs.

Error: Such a protocol is said to be a δ -error protocol, if for all inputs $x \in \{0,1\}^{V(G)}$, $\Pr[\text{output} = f(x)] \geq 1 - \delta$. Here the probability is over the noise in the communication channel as well as the internal randomness, if any, of the protocol.

In this paper, we consider networks that arise out of random placement of sensors in the unit square.

Definition 2 (Random planar network). A random planar network $\mathcal{N}(N,R)$ is a random variable whose values are undirected graphs. The distribution of the random variable depends on two parameters: N, the number of vertices, and R, the transmission radius. The vertex set of $\mathcal{N}(N,R)$ is $V(\mathcal{N}) = 0$

 $\{P_1, P_2, \dots, P_N\}$. The edges are determined as follows. First, these vertices are independently placed at random, uniformly in the unit square $[0, 1]^2$. Then,

$$E(\mathcal{N}) = \{ (P_i, P_j) : \mathsf{dist}(P_i, P_j) < R \},$$

where $dist(P_i, P_j)$ is the Euclidean distance between vertices P_i and P_j .

The result in this paper is the following.

Theorem 3 (Lower bound for parity). Let $R \leq N^{-\beta}$ for some $\beta > 0$. Let $\delta < \frac{1}{2}$ and $\epsilon \in (0,1)$. Then, with probability 1 - o(1) over the random variable $\mathcal{N}(N,R)$, every δ -error protocol on $\mathcal{N}(N,R)$ with ϵ -noise for computing the parity function $\oplus : \{0,1\}^{V(\mathcal{N})} \to \{+1,-1\}$ requires $\Omega(N \log \log N)$ transmissions.

Remark 4. Our definition of noise assumes that all transmissions are subjected to noise with probability exactly ϵ . In the literature, other models of error have been considered. Some protocols work even in the weaker model where this probability is at most ϵ . Our lower bound holds for the stronger model with the noise parameter being exactly ϵ , and hence is also applicable to the weaker model.

Remark 5. We require only an upper bound on the transmission radius. However, the result is meaningful only when $R = \Omega(\sqrt{\frac{\log N}{N}})$, for otherwise, with high probability, the network is not connected and cannot be expected to compute any function that depends on all its input bits.

Remark 6. Trivially, this lower bound also holds for computing the sum of the input bits.

1.1 Related work

The most commonly studied noisy communication model allows full broadcasts, that is, all sensors receive all messages (with independent noise). In this model, Gallager [5] considered the problem of collecting all the bits at one sensor, and showed how this could be done using $O(N \log \log N)$ transmissions; this implies the same upper bound for computing any function of the input bits. More recently, in a remarkable result, Goyal, Kindler and Saks [6] showed that Gallager's protocol was the best possible for collecting all the bits. However, they do not present any boolean function for which $\Omega(N \log \log N)$ transmissions are required.

In the full broadcast model, protocols for computing specific functions have also been studied in the literature. Feige and Raghavan [4] presented a protocol with $O(N\log^* N)$ transmissions for computing the OR of N bits; this result was improved by Newman [11], who gave a protocol with O(N) transmissions. For computing threshold functions Kushilevitz and Mansour [10] showed a protocol with O(N) transmissions, assuming that all messages are subject to noise with probability exactly ϵ . Under the same assumption, Goyal, Kindler and Saks [6] showed that the sum of all the bits (and hence all symmetric functions) could be computed with O(N) transmissions.

In this paper we are concerned with networks arising from random placement of sensors, where considerations of power impose stringent limits on the transmission radius. In this model, Ying, Srikant and Dullerud [12] presented a protocol for computing the sum of all the bits as mentioned above. Kanoria and Manjunath [9] gave a protocol that uses O(N) transmissions to compute the OR function. However, no non-trivial lower bound that apply specifically to communication networks with limited transmission radius had appeared in the literature before this work. Subsequent to the initial presentation of this work [2], Dutta and Radhakrishnan [3] showed that the same lower bound of $\Omega(N \log \log N)$ holds for computing a host of boolean functions including the majority function.

1.2 Techniques

We now present an overview of the proof technique used to derive our lower bound. As we explain in more detail in the Section ??, the proof has two parts. The first part is geometric. Since the transmission radius is limited, it is possible to decompose the nodes of the communication network into clusters. The nodes in the interior of each cluster will continue to receive inputs and will be called *input nodes*, but those on the boundary will have their inputs fixed (arbitrarily) and thereby become *auxiliary nodes* that still participate in the protocol by sending and receiving messages. This decomposition of the communication network into clusters ensures that any node can receive transmissions from input nodes of at most one cluster. This allows us to view the protocol as a combination of several subprotocols acting on different clusters and interacting with each other via the auxiliary nodes. This graph theoretic decomposition is based on routine arguments involving the distribution points chosen independently and uniformly at random on the unit square.

The second part of the proof is combinatorial and concerns arguing that the subprotocols acting on different clusters of the decomposed network can be assumed to be independent of each other. This part is not straightforward and we need to revisit the arguments used by Goyal, Kindler and Saks [6] to obtain their lower bounds. A key insight in their proof was that protocols in noisy communication networks could be translated into what they called *Generalized Noisy Decision trees* (gnd trees). We adapt their argument to our setting. For us it is important to ensure that the decomposition of the network (which was the consequence of the limited transmission radius) is reflected in the noisy decision trees we construct. So, we define a notion of noisy decision trees appropriate for our setting, where we allow each node of the tree access to the inputs of only one cluster. We show how efficient protocols on decomposed networks can be translated to such decision trees of small depth.

The argument this far was general and did not use the fact that the ultimate goal of the protocol is to compute the parity function. Next we show that we can rearrange the decision tree so that the queries made to the variables in the same cluster of the decomposition appear at adjacent levels of the tree. This part crucially depends on the fact that we are trying to compute the parity function. After the rearrangement, we can view the entire computation as a sequence of noisy decision tree computations, one for each cluster. We conclude that in order to have low overall error, the computation in each cluster must have vanishingly small error probability. At this stage we can directly apply a result of Goyal, Kindler and Saks [6], which states that any decision tree that computes the parity function with error o(1) must have superlinear depth. This dependence of depth on error is strong enough to yield our lower bound.

The interesting feature of this argument is that we work with appropriately defined decision trees instead of directly with the decomposed protocol. Once inputs of processors have been fixed, they become auxiliary. However, they continue to participate in the protocol. In particular, they receive transmissions from processors with inputs and can potentially aid error correction by providing additional reception diversity, which is crucially exploited in many of the upper bounds. So it is not true that our decomposition immediately breaks the protocol into independent subprotocols, operating separately on different clusters. Nevertheless, when we translate the decomposed protocol into our model of decision trees, we can view the computation of the entire decision tree as a combination of independent decision subtrees, operating separately on different clusters. This provides us the required product property, from which one easily deduces that each individual subtree must compute the parity within its cluster very accurately. For an detailed discussion of this technique as well as those developed to analyze functions where we do not have the product property, we refer the reader to the Phd thesis [1].

1.3 Organization of the paper

Section 2 presents some definitions and notations. In Section 3, we state two lemmas corresponding to the two parts of the argument, and derive the lower bound for parity. The details of the first part of the argument are presented in Section 4. The second part of the argument is spread over Sections 5 and 6. We conclude the paper in Section 7.

2 Preliminaries

In our proof, some of the nodes in the network will receive no input. We now introduce the terminology applicable in such situations.

Definition 7 (Input and auxiliary nodes). Let G = (V, E) be a communication network. We partition the set of nodes, V, into the set of input nodes, I, and the set of auxiliary nodes, A. Nodes in I receive inputs and those in A do not receive any input but have their input bits fixed arbitrarily. An input to such a network is an element of $\{0,1\}^I$ and a protocol on such a network computes a function $f:\{0,1\}^I \to \{0,1\}$.

Next we formalize the notions of network decomposition and bounded protocols on such decomposed networks.

Definition 8 (Network decomposition and bounded protocols). Let $G = (I \cup A, E)$ be a communication network. An (n, k)-decomposition of G is a partition of the set of nodes of G of the form $I = I_1 \cup \cdots \cup I_k$ and $A = A_0 \cup A_1 \cup \cdots \cup A_k$ such that for $j = 1, \ldots, k$,

- (P1) $|I_j| = n$, and
- (P2) the neighborhood of I_j is contained in $I_j \cup A_j$.

A protocol Π on G is said be a (d, D)-bounded protocol with respect to the decomposition $\langle A_0, (I_j, A_j) : j = 1, ..., k \rangle$ if for j = 1, ..., k,

- (P3) a node in I_i makes at most d transmissions, and
- (P4) all nodes in $I_i \cup A_i$ put together make at most D transmissions.

We use the notation ϵ -noise (n, k, d, D)-protocol to mean a (d, D)-bounded protocol for some (n, k)-decomposed network with noise parameter ϵ .

As stated earlier, we will use the method of Goyal, Kindler and Saks [6] to translate a communication protocol into a noisy decision tree. We now present the terminology for noisy decision trees.

Definition 9 (Decision tree). Let S be an arbitrary set and k be a positive integer. A decision tree \mathcal{T} for the set of inputs S^k is a balanced tree where each internal node v is labelled by a pair $\langle i_v, g_v \rangle$ where $i_v \in [k], g_v : S \to C_v$, and C_v is the set of children of v. We call the tree to be a **noisy** decision tree if the functions g_v are noisy. A noisy function is one whose output depends on its input as well as some internal randomness. Such a tree \mathcal{T} computes a function from S^k to the set $L(\mathcal{T})$ of leaves of \mathcal{T} as follows: on input $\langle x_1, x_2, \ldots, x_k \rangle \in S^k$, the computation starts at the root and determines the next vertex to visit after a vertex v by evaluating $g_v(x_{i_v})$; the leaf reached in the end is the result of the computation. If a vertex $i_v = i$ for a vertex v, then we say that the i-th input variable is queried at that vertex. We say that the decision tree is **oblivious** if the label i_v of a vertex v depends only on the level of v (distance from the root). We say

that an oblivious decision tree is **ordered** if for all $j \in [k]$ all queries to the p-th input variable appear at consecutive levels. We say that an oblivious decision tree is **read-once** if each input variable is queried exactly once.

Remark 10. We use the notation (n,k)-decision tree to refer to a decision tree for inputs in S^k where $S = \{0,1\}^n$.

Remark 11. A read-once decision tree is obviously ordered. Also, an ordered decision tree can be easily made read-once by collapsing consecutive queries to the same variable into one supernode.

As in [6], in order to capture the noise in a noisy communication network, we define a special kind of noisy decision tree, *Xored-Noise Decision tree* (xnd-tree). Here we allow each of the the functions g_v access to its input variable xored with some noise variable. These noise variables are set according to some distribution based on a noise parameter ϵ , but independent of the input.

Definition 12 (xnd tree). An (n, k, D, ϵ) – xnd tree \hat{T} is an (n, k)-noisy decision tree. It consists of an oblivious decision tree T on inputs S^k where $S = \{0, 1\}^n \times (\{0, 1\}^n)^{|\Lambda|}$ (for some index set Λ), and each function g_v has a special form:

$$g_v(x_{i_v}, \overline{z}_{i_v}) = g'_v(x_{i_v} \oplus z_{i_v, \lambda_v}),$$

for some $g'_v: \{0,1\}^n \to C_v$ and $\lambda_v \in \Lambda$. Each input is queried at most D times in the tree. The computation of $\hat{\mathcal{T}}$ proceeds as follows: on input $x \in (\{0,1\}^n)^k$, each $z_{i,\lambda} \in \{0,1\}^n$ is chosen independently according to the binomial distribution $\mathcal{B}(n,\epsilon)$. Once the entire input $(\overline{x},\overline{z}) \in S^k$ is determined, we compute $\mathcal{T}(\overline{x},\overline{z})$ as in Definition 9 above.

Remark 13. When k = 1, the trees defined in the above definition correspond to the gnd trees of Goyal, Kindler and Saks [6].

Let \mathcal{A} be an algorithm to process inputs from some set S. The usefulness of \mathcal{A} to compute some boolean function f on input set S is captured by the notion of its *advantage*.

Definition 14 (Advantage). Let μ be a distribution on some set S. Let $f: S \to \{+1, -1\}$ and $A: S \to C$, where C is some set. Then, the advantage of A for f under μ is given by

$$\mathsf{adv}_{f,\mu}(\mathcal{A}) = \max_{a:C \to [-1,+1]} |\operatorname{E}[f(X)a(\mathcal{A}(X))]|,$$

where X is a random variable taking values in S with distribution μ . We will use this notation even when A corresponds to a randomized algorithm, in which case, the expectation is computed over X as well as the internal random choices made by A.

Definition 15. For a distribution μ on $\{0,1\}^n$, let

$$\alpha_{\mu}(n, D, \epsilon) \stackrel{\Delta}{=} \max_{T} \mathsf{adv}_{\oplus, \mu}(T),$$

where T ranges over all $(n, 1, D, \epsilon)$ – xnd trees.

3 Lower bound for parity

Our lower bound proof has two parts. In this section, we will summarize the results of these two parts of the argument in the form of lemmas. Then, using these lemmas we will prove the main theorem. The lemmas themselves will be proved in the next three sections.

3.1 First part of the proof

This part of our argument is based on the observation that in a random planar network, nodes are typically distributed uniformly over the entire area. By fixing the inputs of some of the nodes (and thereby making them auxiliary), we can create 'buffer zones' of auxiliary nodes so that the remaining nodes now fall into large number of well-separated large clusters.

Lemma 16. Suppose $R \leq N^{-\beta}$, for some $\beta > 0$. Then, with probability 1 - o(1) over the random variable $\mathcal{N}(N, R)$, the following holds: if

there is a δ -error protocol on N with ϵ -noise for computing the parity function (on N bits) with T transmissions,

then

there is an (n, k)-decomposition of N and a δ -error ϵ -noise (n, k, d, D)-protocol with respect to this decomposition for computing parity (on nk bits), where $n = \Omega(NR^2)$, $k = \Omega(1/R^2)$, d = O(T/N) and $D = O(TR^2)$.

This lemma is proved in Section ??.

3.2 second part of the proof

In the second part of our argument, we analyze such bounded protocols on decomposed networks. Our analysis closely follows that of Goyal, Kindler and Saks [6]. For showing lower bounds on the number of transmissions in a noisy communication protocol, Goyal et al. translated such protocols into gnd trees.

since we want to analyse bounded protocols for decomposed networks, we first translate such protocols into xnd-trees. Then we argue that if the inputs come from a product distribution, then xnd-trees for computing parity can be rearranged to get ordered xnd-trees, and hence read-once noisy decision trees (using Remark 11).

Lemma 17 (Translation from protocols to read-once decision trees). For any ϵ -noise (n, k, d, D)-protocol Π and any distribution μ on $\{0, 1\}^n$, there is a read-once noisy (n, k)-decision tree \mathcal{T} such that

- $\operatorname{\mathsf{adv}}_{\oplus,\mu^k}(\mathcal{T}) \geq \operatorname{\mathsf{adv}}_{\oplus,\mu^k}(\Pi)$;
- $adv_{\oplus,\mu}(g) \leq \alpha_{\mu}(n,3D,\epsilon^d)$ for every function g that appears in \mathcal{T} .

Next we observe the following 'product property' for the advantage of read-once noisy decision trees.

Lemma 18 (Advantage of read-once decision trees). Let $h:\{0,1\}^n \to \{+1,-1\}$. Suppose \mathcal{T} is a read-once (n,k)-decision tree for computing $f:(\{0,1\}^n)^k \to \{+1,-1\}$ defined by $f(\langle x_1,x_2,\ldots,x_k\rangle) = \prod_{i=1}^k h(x_i)$. Suppose, for each function $\mathcal A$ that appears in $\mathcal T$ we have $\mathsf{adv}_{h,\mu}(g) \le \alpha$. Then, $\mathsf{adv}_{f,\mu^k}(\mathcal T) \le \alpha^k$.

The above two lemmas give the main lemma of the second part of our proof.

Lemma 19. For all distributions μ on $\{0,1\}^n$ and all ϵ -noise (n,k,d,D)-protocol Π , we have

$$\mathsf{adv}_{\oplus,\mu^k}(\Pi) \leq \alpha_\mu(n,3D,\epsilon^d)^k.$$

Proof. Immediate from Lemma 17 and Lemma 18.

Section ?? is devoted to proving Lemma 17, and Section ?? proves Lemma 18.

3.3 Putting the two parts together

To complete the proof of our lower bound, we need the following result of [6].

Definition 20. Let $f: \{0,1\}^n \to \{0,1\}$ be any function. The sensitivity of f at input $x \in \{0,1\}^n$, denoted $S_x(f)$, is the number of indices $i \in [n]$ such that f changes value upon flipping the ith bit of x. The sensitivity of f, denoted s(f), is the maximum of $S_x(f)$ over all x.

Theorem 21 (Goyal, Kindler and Saks [6] (Theorem 32)). Let $\epsilon \in (0, 1/2)$ and $\delta \in (0, 1/16)$, and let f be an n-variate boolean function. Any randomized gnd tree T that for every input x, outputs f(x) with probability $1 - \delta$ when run with noise parameter ϵ satisfies:

$$\operatorname{depth}(T) \geq \frac{\epsilon^2 \log(1/4\delta)}{50 \log^2(1/\epsilon)} s(f).$$

We will restate the above theorem for the case of parity in terms of advantage of xnd trees.

Theorem 22 (Restatement of Theorem 21). Let μ be the distribution on $\{0,1\}^n$ defined by $\mu(0^n) = \frac{1}{2}$ and $\mu(e) = \frac{1}{2n}$ for all $e \in \{0,1\}^n$ of weight 1. Then

$$\alpha_{\mu}(n, D, \epsilon) \le \max\left(1 - \exp\left(-O\left(\frac{D\log^2(1/\epsilon)}{\epsilon^2 n}\right)\right), 7/8\right).$$
 (1)

Proof of the restatement. Let μ be as given in the theorem. Theorem 21 is proved in [6] by proving an upper bound on the probability that T is correct when T is executed on an input selected at random from the distribution μ . Thus any gnd tree T that makes an average error of at most $\delta < 1/16$ for computing the parity function $\oplus : \{0,1\}^n \to \{0,1\}$ on inputs from the distribution μ , when run with noise parameter ϵ , must have

$$\operatorname{depth}(T) \geq \frac{\epsilon^2 \log(1/4\delta)}{50 \log^2(1/\epsilon)} n,$$

since the sensitivity of the parity function \oplus : $\{0,1\}^n \to \{0,1\}$ is n. As the RHS of the above equation is strictly decreasing with δ , we conclude that any $(n,1,D,\epsilon)$ — xnd tree T makes an average error of at least δ' for computing the parity function on inputs from the distribution μ , where

$$\delta' = \min\left(\exp\left(-O\left(\frac{\log^2(1/\epsilon)D}{\epsilon^2 n}\right)\right), 1/16\right).$$

Thus $\mathsf{adv}_{\oplus,\mu}(T) \leq 1 - 2\delta'$, which proves the theorem.

Proof of Theorem 3. Let μ be the distribution defined in Theorem 22. By combining Lemmas 16 and 19, we conclude that with probability 1 - o(1) over the random variable $\mathcal{N}(N, R)$, the following is true: if there is a δ -error protocol on $\mathcal{N}(N, R)$ with ϵ -noise for computing the parity function with T transmissions, then

$$1 - 2\delta \le \alpha_{\mu}(n, 3D, \epsilon^d)^k,$$

where $n=\Omega(NR^2)$, $k=\Omega(1/R^2)$, d=O(T/N) and $D=O(TR^2)$.

Since $R \leq N^{-\beta}$, $k = \Omega(1/R^2)$ and δ is a constant, $\alpha_{\mu}(n, 3D, \epsilon^d)$ must be inverse polynomially close to 1. Let $k \geq C/R^2$ and $d \leq C'T/N$ for some constants C, C'. From (1), we thus get

$$1 - 2\delta \le \left(1 - \exp\left(-O\left(\frac{TR^2\log^2(1/\epsilon^{C'T/N})}{NR^2\epsilon^{2C'T/N}}\right)\right)\right)^{\frac{C}{R^2}}.$$

Denoting T/N by S and simplifying, we have

$$1 - 2\delta \le \exp\left(-\exp\left(-O\left(\frac{S\log^2(1/\epsilon^{C'S})}{\epsilon^{2C'S}}\right)\right)\frac{C}{R^2}\right).$$

Taking logarithm and noting that $R \leq N^{-\beta}$,

$$\exp\left(-O\left(\frac{S\log^2(1/\epsilon^{C'S})}{\epsilon^{2C'S}}\right)\right) \leq \frac{N^{-2\beta}}{C}\ln\left(\frac{1}{1-2\delta}\right).$$

From this we get,

$$\frac{S \log^2(1/\epsilon^{C'S})}{\epsilon^{2C'S}} \ge C'' \log N,$$

for some constant C''. This yields $S = \Omega(\log \log N)$ and hence $T = \Omega(N \log \log N)$.

4 Decomposition of random planar networks

The random placement of nodes in the unit square typically arranges them uniformly. We will exploit this uniformity to obtain the required decomposition.

Lemma 23 (Chernoff bounds). Let X be the sum of N independent identically distributed indicator random variables. Let $\mu = E[X]$. Then, $\Pr[X \le \frac{1}{2}\mu] \le \exp(-0.15\mu)$.

Proof. The lemma follows immediately from the following version of the Chernoff bound due to Hoeffding [8]: if the random variable X has binomial distribution $\mathcal{B}(N, p)$, then

$$\Pr[X \ge (p+\delta)N] \le \left(\frac{p}{p+\delta}\right)^{(p+\delta)N} \left(\frac{1-p}{1-p-\delta}\right)^{(1-p-\delta)N}.$$
 (2)

To derive the lemma, we consider the random variable Y=N-X, and apply (2) with $p=1-\frac{\mu}{N}$ and $\delta=\frac{\mu}{2N}$, to obtain

$$\begin{aligned} \Pr[X \leq \frac{1}{2}\mu] &\leq \Pr[Y \geq (p+\delta)N] \\ &\leq \left(1 - \frac{\delta}{p+\delta}\right)^{(p+\delta)N} \!\!\! \left(\frac{1-p}{1-p-\delta}\right)^{(1-p-\delta)N} \\ &\leq \exp(-\delta N) \cdot 2^{\frac{\mu}{2}} \\ &\leq \exp\left(-\frac{1}{2}(1-\ln 2)\mu\right) \\ &\leq \exp(-0.15\mu). \end{aligned}$$

Proof of lemma 16. We tessellate the unit square into $M = (\lfloor 1/R \rfloor)^2$ cells, each a square of side $\frac{1}{\lfloor 1/R \rfloor}$. We number the rows and columns of this tessellation using indices in $\{1, 2, \ldots, \lfloor 1/R \rfloor\}$, and refer to the cell in the *i*-th row and *j*-th column by c_{ij} . The expected number of processors in any one cell is $\mu = N/M$. Since

 $R \ge \sqrt{10 \ln N/N}$, we have $\mu \ge 10 \ln N$, and by Lemma 23, the probability that there are fewer than $\mu/2$ processors in any one cell is is $o(\frac{1}{M})$. So, with probability 1-o(1), all cells have at least $\mu/2 = N/(2M)$ processors.

Now, let $S_1 = \{c_{ij} : i = 1 \pmod 3 \text{ and } j = 1 \pmod 3 \}$. Then, $|S_1| \ge M/9$. For each $c \in S_1$, let the neighborhood of c, denoted by $\Gamma(c)$, be the set of (at most nine) cells that are at distance less than R from c. Note that distinct cells in S_1 have disjoint neighborhoods. If the total number of transmissions in the original protocol is T, then the average number of transmissions made from $\Gamma(c)$ as c ranges over S_1 is at most 9T/M. By Markov's inequality, for at least half the cells $c \in S_1$ fewer than 18T/M transmissions are made from $\Gamma(c)$. Let S_2 be the set of these cells; $|S_2| \ge M/18$. For each cell $c \in S_2$, we identify the set I_c of $\lceil N/(4M) \rceil$ processors that make fewest transmissions. We are now ready to describe the decomposition of the planar communication network.

The set of input processors will be $I = \bigcup_{c \in \mathcal{S}_2} I_c$. We fix the input of all processors not in I at 0, and treat them as auxiliary processors. The protocol continues to compute the parity of the inputs provided to processors in I. For $c \in \mathcal{S}_2$, let A_c be the set of auxiliary processors in the cells in $\Gamma(c)$. Also let A_0 be the set of all those auxiliary processors that are not in $\Gamma(c)$ for any $c \in \mathcal{S}_2$. We have thus obtained a decomposition $\langle A_0, (I_c, A_c) : c \in \mathcal{S}_2 \rangle$, such that

- (a) the number of input classes in the decomposition is $k = |S_2| \ge M/18$;
- (b) each input class has $n = \lceil \mu/4 \rceil$ processors;
- (c) The total number of transmissions made by all processors in $I_c \cup A_c$ is at most D = 18T/M;
- (d) The total number of transmissions made by any one processor in I_c is at most d = D/n = 72T/N.

Thus we have obtained an (n,k)-decomposition of the network \mathcal{N} and the original protocol now reduces to a δ -error ϵ -noise (n,k,d,D)-protocol with respect to this decomposition for computing the parity function on nk bits, where $n \geq NR^2/4$, $k \geq \frac{1}{18} \left\lfloor 1/R \right\rfloor^2$, $d \leq 72T/N$ and $D \leq 18TR^2$.

5 Translation from protocols to read-once decision trees

In this section, we will first translate bounded protocols for decomposed networks into xnd trees. Then we will show how we can rearrange oblivious decision trees in some cases to make them ordered. These two steps will then enable us to prove lemma 17.

5.1 From bounded protocols to xnd trees

Lemma 24. For any ϵ -noise (n, k, d, D)-protocol Π and any distribution μ on $(\{0, 1\}^n)^k$, there is an $(n, k, 3D, \epsilon^d)$ -xnd tree \mathcal{T} such that $\mathsf{adv}_{\oplus, \mu}(\mathcal{T}) \geq \mathsf{adv}_{\oplus, \mu}(\Pi)$.

Proof. We will carry out the translation from bounded protocols to xnd trees via two intermediate models of communication protocols.

Definition 25 (Intermediate protocols). The following two kinds of protocols are obtained by imposing restrictions on bounded protocols for decomposed networks of Definition 8.

Semi-noisy protocol: An ϵ -noise (n, k, d, D)-semi-noisy protocol differs from an ϵ -noise (n, k, d, D)-protocol only in the following respects.

- (a) When it is the turn of an input processor to send a message, it sends only its input bit, whose independent ϵ -noisy copies are then received by its neighbors.
- (b) A transmission made by an auxiliary processor is not subjected to any noise.

Noisy copy protocol: An ϵ -noise (n, k, D)-noisy-copy protocol is an ϵ -noise (n, k, 1, D)-semi-noisy protocol; in other words, every input processor makes exactly one broadcast of its input bit, so that each of its neighbors receives exactly one independent ϵ -noisy copy of this input bit.

Remark 26. In these special kinds of protocols, the messages sent by the input processors does not depend on the messages these processors receive. Thus, we may assume that the input processors make their transmissions in the beginning of the protocol an appropriate number of times, and after that the auxiliary processors interact according to a zero noise protocol.

Claim 27 (From bounded protocol to semi-noisy). For every function $f:(\{0,1\}^n)^k \to \{+1,-1\}$, distribution μ on $(\{0,1\}^n)^k$ and every ϵ -noise (n,k,d,D)-protocol Π , there is an ϵ -noise (n,k,d,3D)-semi-noisy protocol Π_1 such that $\mathsf{adv}_{f,\mu}(\Pi) \leq \mathsf{adv}_{f,\mu}(\Pi_1)$.

Claim 28 (From semi-noisy to noisy-copy). For every function $f:(\{0,1\}^n)^k \to \{+1,-1\}$, distribution μ on $(\{0,1\}^n)^k$ and every ϵ -noise (n,k,d,D)-semi-noisy protocol Π_1 , there is an ϵ^d -noise (n,k,D)-noisy-copy protocol Π_2 such that $\mathsf{adv}_{f,\mu}(\Pi_1) \leq \mathsf{adv}_{f,\mu}(\Pi_2)$.

Claim 29 (From noisy-copy to xnd tree). For every function $f: (\{0,1\}^n)^k \to \{+1,-1\}$, distribution μ on $(\{0,1\}^n)^k$ and every ϵ -noise (n,k,D)-noisy-copy protocol Π_2 , there is an (n,k,D,ϵ) -xnd tree $\mathcal T$ such that $\mathsf{adv}_{f,\mu}(\Pi_2) \le \mathsf{adv}_{f,\mu}(\mathcal T)$.

Lemma 24 follows immediately from Claims 27, 28 and 29.

Proof of Claim 27. Fix an ϵ -noise (n,k,d,D)-protocol Π on a graph G. We will construct an ϵ -noise (n,k,d,3D)-semi-noisy protocol Π_1 on a graph $G_1=(V_1,E_1)$. The graph G_1 will contain G as a subgraph; however, all vertices inherited from G will correspond to auxiliary processors. In addition, for each input vertex v of G, we will have a new input vertex v' in G_1 , which will be connected to v and its neighbors in G. Let $(I=\bigcup_{j=1}^k I_j, A=A_0\cup\bigcup_{j=1}^k A_j)$ be the decomposition corresponding to Π . The decomposition corresponding to Π_1 will be $(I'=\bigcup_{j=1}^k I'_j, A'=A_0\cup\bigcup_{j=1}^k A'_j)$, where $I'_j=\{v':v\in I_j\}$ and $A'_j=A_j\cup I_j$.

Suppose Π uses T transmissions. For $i=1,2,\ldots,T$ and $v\in V(G)$, let $b_v[i]$ be the bit received by v when the i-th transmission is made; if v does not receive the i-th transmission, we define $b_v[i]$ to be 0. The protocol Π_1 for simulating Π will operate in T stages, one for each transmission made by Π . The goal is to ensure that in the end each auxiliary processor v of G_1 constructs a sequence $b_v' \in \{0,1\}^T$, such that $\langle b_v' : v \in V(G) \rangle$ and $\langle b_v : v \in V(G) \rangle$ (of the protocol Π) have the same distribution, for every input in $(\{0,1\}^n)^k$. This implies that the outputs of Π' and Π have the same distribution. Suppose the first $\ell-1$ stages have been successfully simulated and $\langle b_v'[1,\ldots,\ell-1] : v \in V(G) \rangle$ have been appropriately constructed. We now describe how stage ℓ is implemented and $\langle b_v'[\ell] : v \in V(G) \rangle$ are constructed. If the ℓ -th transmission in Π is made by an auxiliary processor v in G, then it will be simulated in Π_1 using one noiseless transmission from v; if the ℓ -th transmission is made by an input vertex v of G, then it will be simulated in Π_1 using two (noiseless) transmissions from v and one ϵ -noisy transmission from the corresponding (newly added) input vertex v'.

v is an auxiliary vertex in G: The auxiliary vertex v in G_1 operates exactly in the same fashion as in G, and sends a bit b, which is received without error by all its neighbors. Each neighbor $w \in V(G)$ of v independently sets its bit $b'_w[\ell]$ to be an ϵ -noisy copy of b (using its internal randomness).

v is an input vertex in G: The auxiliary vertex v in G_1 has all the information that the corresponding input vertex v in G would have had, except the input (which is now given to the new input vertex v'). So, v transmits (with no noise) two bits, b_0 and b_1 , corresponding to the two possible input values that v' might have. Next, the input vertex v' transmits its input c; let c_w denote the ϵ -noisy version of c that the neighbor $w \in V(G)$ receives. Each neighbor w of v now acts as follows: if $b_0 = b_1$, then it sets $b'_w[\ell]$ to be an ϵ -noisy copy of b_0 (using its internal randomness); if $b_0 \neq b_1$, then it sets $b'_w[\ell]$ to b_{c_w} .

Proof of Claim 28. Let Π_1 be an ϵ -noise (n,k,d,D)-semi-noisy protocol. As remarked above, all input processors in a semi-noisy protocol can be assumed to make their transmissions right in the beginning, after which only the auxiliary processors operate. Thus, each auxiliary processor receives at most d independent ϵ -noisy copies of the input from each input processor in its neighborhood. The following lemma of Goyal, Kindler and Saks [6] shows that a processor can generate d independent ϵ -noisy copies of any input from one ϵ^d -noisy copy.

Lemma 30 (Goyal, Kindler and saks [6] (Lemma 36)). Let t be an arbitrary integer, $\epsilon \in (0, 1/2)$ and $\gamma = \epsilon^t$. There is a randomized algorithm that takes as input a single bit b and outputs a sequence of t bits and has the property that if the input is a γ -noisy copy of 0 (respectively of 1), then the output is a sequence of independent ϵ -noisy copies of 0 (respectively of 1).

We modify the protocol Π_1 to an ϵ^d -noise (n,k,D)-noisy-copy protocol Π_2 by requiring that each input processor makes one ϵ^d -noisy transmission of its input bit. Each auxiliary processor on receiving such a transmission uses its internal randomness to extract the required ϵ -noisy copies. Then onwards the protocol proceeds as before. We may now fix internal randomness used by the auxiliary processors in such a way that the advantage of the resulting protocol for the input distribution μ is at least as good as that of the original protocol. Thus, all processors use (deterministic) functions to compute the bit that they transmit.

Proof of Claim 29. Let Π_2 be an ϵ -noise (n,k,D)-noisy-copy protocol, with the underlying decomposition $(I=\bigcup_{j=1}^k I_j, A=A_0\cup\bigcup_{j=1}^k A_j)$. We will now show how this protocol can be simulated using an (n,k,D,ϵ) -xnd tree \mathcal{T} . To keep our notation simple, we will assume (by introducing new edges, if necessary) that (a) all processors in A are adjacent, and (b) every processor in A_j is adjacent to every processor in I_j .

Let T be the total number of transmissions in Π_2 . Let b_1, b_2, \ldots, b_T be the sequence of bits transmitted in Π_2 by the auxiliary processors. Suppose, b_i is transmitted by vertex $v \in I_j$ by computing $g_i(b_1b_2\cdots b_{i-1},x_j\oplus z_v)$, where x_j is the the restriction of the input assignment to I_j and z_v is an ϵ -noisy vector in $\{0,1\}^n$.

The nodes of the xnd tree \mathcal{T} are 0-1 sequences of length at most T (the root is the node at 0th level and corresponds to the empty sequence). The children of the node $b \in \{0,1\}^{i-1}$ ($0 \le i-1 \le T-1$) are the two vertices b0 and b1. Suppose vertex $v \in A_j$ makes the i-th transmission. The function that v computes to determine what to transmit, will be used to compute the successor of the nodes at the i-1-th level. To state this formally, the label of $b \in \{0,1\}^{i-1}$ (at level i-1 in \mathcal{T}) is (j,h), where $h(x_j,z_v)=b\cdot g_i(b,x_j\oplus z_v)$. (Since our definition requires the function to return a child of b, h returns an extension of b in $\{0,1\}^i$.)

The set of leaves of \mathcal{T} , $L(\mathcal{T})$, is precisely $\{0,1\}^T$. Let $a:L(\mathcal{T})\to\{+1,-1\}$ be defined by $a(b_1b_2\cdots b_T)=(-1)^{b_T}$. Then, it follows from our definitions that

$$\begin{array}{lll} \mathsf{adv}_{\oplus,\mu}(\mathcal{T}) & \geq & |\operatorname{E}[\oplus(x)a(\mathcal{T}(x))]| \\ & = & |\operatorname{E}[\oplus(x)(-1)^{b_T}]| \\ & = & \mathsf{adv}_{\oplus,\mu}(\Pi_2). \end{array}$$

5.2 Tree rearrangement

Our main observation in this section is that oblivious decision trees can be assumed to be ordered when the inputs come from a product distribution, and we wish to approximate the parity function. To show this we will describe a method for rearranging an arbitrary oblivious decision tree so that it becomes ordered.

Definition 31 (Tree rearrangement). Let \mathcal{T} and \mathcal{T}' be oblivious decision trees for the same set of inputs. We say that \mathcal{T}' is a rearrangement of tree \mathcal{T} if

- both trees query each variable the same number of times;
- the functions labelling vertices of \mathcal{T}' also appear in \mathcal{T} (up to obvious renaming of children); formally, for every vertex \hat{v} in \mathcal{T}' labelled (i, \hat{g}) , there is a vertex v in \mathcal{T} labelled (i, g) in \mathcal{T} and a bijection $\pi : C_{\hat{v}} \to C_v$ such that $\forall x \in S_i : \hat{g}(x) = \pi(g(x))$.

Lemma 32 (Ordering lemma). Let μ be a product distribution on some set S^k . Let $f: S^k \to \{+1, -1\}$ be of the form $f(x_1, x_2, \dots, x_k) = h(x_1)h(x_2)\cdots h(x_k)$, where $h: S \to \{+1, -1\}$. Then every oblivious decision tree \mathcal{T} can be rearranged to obtain an ordered oblivious decision tree $\hat{\mathcal{T}}$ such that $\mathsf{adv}_{f,\mu}(\hat{\mathcal{T}}) \ge \mathsf{adv}_{f,\mu}(\mathcal{T})$.

This lemma will follow immediately from the following claim.

Claim 33 (Move to root). Let μ be a product distribution on S^k . Let $f: S^k \to \{+1, -1\}$ be of the form $f(x_1, x_2, \dots, x_k) = h(x_1)h(x_2)\cdots h(x_k)$, where $h: S \to \{+1, -1\}$. Let \mathcal{T} be an oblivious decision tree with inputs in S^k such that the input x_n is queried only at the level just above the leaves. Then, \mathcal{T} can be rearranged to obtain a tree $\hat{\mathcal{T}}$ where

- 1. the input x_k is queried only at the root;
- 2. for all $j \neq k$, if x_j was queried at level r of T, then x_j is queried at level r + 1 of \hat{T} ;
- $3. \ \operatorname{\mathsf{adv}}_{f,\mu}(\hat{\mathcal{T}}) \geq \operatorname{\mathsf{adv}}_{f,\mu}(\mathcal{T}).$

Proof. Let $X = \langle X_1, X_2, \dots, X_k \rangle$ take values in S^k with distribution μ ; since μ is a product distribution the X_i 's are independent. Suppose \mathcal{T} makes t queries to the input. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{t+1}$ be the random sequence of vertices visited by the computation of \mathcal{T} on input X. Fix $b: L(\mathcal{T}) \to [-1, +1]$ such that

$$\mathsf{adv}_{f,\mu}(\mathcal{T}) = |\operatorname{E}[h(X_1)h(X_2)\cdots h(X_k)b(\mathbf{v}_{t+1})]|$$
$$= |\operatorname{E}[\operatorname{E}[h(X_1)\cdots h(X_k)b(q_{\mathbf{v}_t}(X_k)) \mid \mathbf{v}_t]]|.$$

Since X_k is queried only at the end, $h(X_1) \dots h(X_{k-1})$ and $b(g_{\mathbf{v}_t}(X_k))$ are independent given \mathbf{v}_t , so $\mathrm{E}[h(X_1) \dots h(X_{k-1}) h(X_k) b(g_{\mathbf{v}_t}(X_k)) \mid \mathbf{v}_t] = \mathrm{E}[h(X_1) \dots h(X_{k-1}) \mid \mathbf{v}_t] \cdot \mathrm{E}[h(X_k) b(g_{\mathbf{v}_t}(X_k)) \mid \mathbf{v}_t]$.

Let $\alpha(v) = \mathrm{E}[h(X_1) \dots h(X_{k-1}) \mid \mathbf{v}_t = v]$ and $\beta(v) = \mathrm{E}[h(X_k)b(g_{\mathbf{v}_t}(X_k)) \mid \mathbf{v}_t = v]$. Let $v^* = \arg\max\beta(v)$; thus, among the functions labelling vertices that query X_k (at level t), g_{v^*} has the best advantage in the tree for h under the distribution of X_k . It is thus natural to expect (and not hard to verify) that if we replace all queries to X_k by this query g_{v^*} , the overall advantage can only improve. Once this is done, the last query does not depend on the previous query, and can, therefore, be moved to the root. We now present the argument formally. We have,

$$\mathsf{adv}_{f,\mu}(\mathcal{T}) = |\operatorname{E}[\alpha(\mathbf{v}_t)\beta(\mathbf{v}_t)]|$$

$$\leq \operatorname{E}[|\alpha(v_T)|] \cdot |\beta(v^*)|.$$
(3)

We are now ready to describe the rearrangement of \mathcal{T} . Let \mathcal{T}^- be the subtree of \mathcal{T} consisting of the first t levels of vertices; thus vertices where X_k is queried in \mathcal{T} become leaves in \mathcal{T}^- . We first make $|C_{v^*}|$ copies of \mathcal{T}^- ; we refer to these copies by \mathcal{T}_c^- ($c \in C_{v^*}$), and assume that the root of \mathcal{T}_c^- is renamed c. In the new tree $\hat{\mathcal{T}}$, we have a root with label $\langle k, g_{v^*} \rangle$ which is connected to the subtrees \mathcal{T}^- . We claim that $\mathsf{adv}_{f,\mu}(\hat{\mathcal{T}}) \geq \mathsf{adv}_{f,\mu}(\mathcal{T})$. Indeed, consider the function $\hat{b}: L(\hat{\mathcal{T}}) \to [-1,+1]$ that takes the value $\mathsf{sign}(\alpha(v))b(c)$ on the leaf in \mathcal{T}_c^- corresponding to $v \in L(\mathcal{T}^-)$. Then, we have

$$\mathsf{adv}_{f,\mu}(\hat{\mathcal{T}}) \geq |\operatorname{E}[h(X_1)h(X_2)\cdots h(X_k)\hat{b}(\hat{v}_T)]|$$

$$= \operatorname{E}[|\alpha(\mathbf{v}_T)|]\cdot |\beta(v^*)|.$$

$$(4)$$

Claim 33 now follows by combining (3) and (4).

We are now ready to show how trees computing the parity function can be reordered, and prove Lemma 32. The argument essentially involves repeated application of Claim 33 to place all queries made to a variable in adjacent levels. We state the argument formally by considering a carefully defined *minimal counterexample*.

Proof of Lemma 32. Fix an oblivious decision tree \mathcal{T} . Let the depth \mathcal{T} be r. We say that there is an alternation at level $\ell \in \{3, \ldots, r\}$ of \mathcal{T} if the variable queried at level ℓ is queried at a level before $\ell-1$ but not at level $\ell-1$. Clearly, a tree with no alternations is an ordered tree. Among all rearrangements of \mathcal{T} , let $\hat{\mathcal{T}}$ be such that

- (P1) $\operatorname{\mathsf{adv}}_{f,\mu}(\hat{\mathcal{T}}) \geq \operatorname{\mathsf{adv}}_{f,\mu}(\mathcal{T});$
- (P2) among all \hat{T} satisfying (P1), \hat{T} has the fewest alternations;
- (P3) among all \hat{T} satisfying (P1) and (P2), the last alternation in \hat{T} is farthest from the root.

We claim that $\hat{\mathcal{T}}$ has no alternations. Let us assume that $\hat{\mathcal{T}}$ has alternations and arrive at a contradiction. Let $\hat{\mathcal{T}}'$ be the tree obtained from $\hat{\mathcal{T}}$ by merging queries on adjacent levels into one *superquery*. That is, if there are j adjacent levels somewhere in the tree that query x_i , with two outcomes, then we replace these j levels by a single superquery with 2^j outcomes. Note that the number of alternations in $\hat{\mathcal{T}}'$ is the same as in $\hat{\mathcal{T}}$. Let r' be the number of queries in $\hat{\mathcal{T}}'$. We consider two cases:

 \mathcal{T}' does not have an alternation at level r': Let x_1 be the variables queried at level r'. By Claim 33, we obtain a tree $\hat{\mathcal{T}}''$ where the superquery to x_1 appears only at the root, and all other superqueries are shifted one level down. Now, however, if each superquery in $\hat{\mathcal{T}}''$ is replaced by its corresponding subtree of queries from $\hat{\mathcal{T}}$, then we obtain a rearrangement of $\hat{\mathcal{T}}$ satisfying (P1) and (P2), but with alternation at a level farther from the root, contradicting (P3).

 \mathcal{T}' has an alternation at level r': Suppose x_1 is queried at level r', and the previous query to x_1 is at level r'' < r' (with no queries to x_1 in the levels $r'' + 1, r'' + 2, \ldots, r' - 1$). Now, we apply Claim 33 to the subtrees of \mathcal{T}' rooted at level r'' + 1, thereby obtaining a rearrangement $\hat{\mathcal{T}}''$, where x_1 is now queried at levels r'' + 1 instead of at level r'. Clearly, the resulting tree $\hat{\mathcal{T}}''$ has fewer alternations than $\hat{\mathcal{T}}'$. Furthermore, if each superquery in $\hat{\mathcal{T}}''$ is replaced by its corresponding tree of queries from $\hat{\mathcal{T}}$, we obtain a rearrangement of $\hat{\mathcal{T}}$. It can be verified that this rearrangement has advantage at least no worse than $\hat{\mathcal{T}}$ but has fewer alternations—contradicting (P2).

5.3 Obtaining the read-once decision tree

Proof of Lemma 17. By combining Lemmas 24 and 32, we see that Π can be converted into an ordered $(n,k,3D,\epsilon^d)$ -xnd tree. Since this tree is ordered all queries to any particular variable appear in consecutive levels. In our final tree we will combine all these queries into a single query. In particular, if there are $\ell \leq 3D$ levels that query (x_i,z_i) , then we collapse them, so as to yield a single query with 2^ℓ outcomes. Note, however, that the result of this query depends not only on the real input in $x_i \in \{0,1\}^n$ but also on the noise variable z_i . In the final noisy decision tree \mathcal{T} , we regard this superquery $g(x_i)$ as a noisy function of the input x_i , with z_i providing the internal randomness for its computation. Since $g(x_i)$ was derived from an $(n,1,\ell,\epsilon^d)$ -xnd tree with $\ell \leq 3D$, it follows from the definition of $\alpha_\mu(n,3D,\epsilon^d)$ that $\mathrm{adv}_{\oplus,\mu}(g) \leq \alpha_\mu(n,3D,\epsilon^d)$.

6 Analysis of read-once decision trees

In this section, we will prove Lemma 18. We will make use of the following proposition.

Proposition 34. Let X be a random variable taking values in $\{0,1\}^n$ with distribution μ . Then, for all $f:\{0,1\}^n \to \{+1,-1\}$, $\mathcal{A}:\{0,1\}^n \to C$ and $a:C\to\mathbb{R}$,

$$|\operatorname{E}[f(X)a(\mathcal{A}(X))]| \le |a| \cdot \operatorname{adv}_{f,\mu}(\mathcal{A}),$$

where $|a| = \max_{c \in C} |a(c)|$.

Proof.

$$\begin{split} |\operatorname{E}[f(X)a(\mathcal{A}(X))]| &= |\sum_{c \in C} \operatorname{E}[f(X)a(\mathcal{A}(X))|\mathcal{A}(X) = c] \cdot \Pr[\mathcal{A}(X) = c]| \\ &\leq \sum_{c \in C} |a(c)| \cdot |\operatorname{E}[f(X)|\mathcal{A}(X) = c]| \cdot \Pr[\mathcal{A}(X) = c] \\ &\leq \max_{c \in C} |a(c)| \cdot \sum_{c \in C} |\operatorname{E}[f(X)|\mathcal{A}(X) = c]| \cdot \Pr[\mathcal{A}(X) = c] \\ &= |a| \cdot \sum_{c \in C} \operatorname{E}[f(X)b(\mathcal{A}(X))|\mathcal{A}(X) = c] \cdot \Pr[\mathcal{A}(X) = c] \\ &\leq |a| \cdot |\operatorname{E}[f(X)b(\mathcal{A}(X))]| \\ &\leq |a| \cdot \operatorname{adv}_{f,\mu}(\mathcal{A}), \end{split}$$

where $b: C \to \{+1, -1\}$ is defined as $b(c) = \text{sign}(\mathbb{E}[f(X)|\mathcal{A}(X) = c])$ for all $c \in C$.

Proof of Lemma 18. Fix $b: L(\mathcal{T}) \to [-1, +1]$. Let X take values in $(\{0, 1\}^n)^k$ with distribution μ^k . We wish to show that

$$|\operatorname{E}[f(X)b(\mathcal{T}(X))]| \le \alpha^k.$$

Let the (random) sequence of vertices visited by the computation of \mathcal{T} on input X be $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}$. For $i = 1, 2, \dots, k$ and v in level i of the tree (at distance i - 1 from the root) let

$$\alpha_i(v) = \mathbb{E}[h(X_i)h(X_{i+1})\cdots h(X_k)b(\mathbf{v}_{k+1}) \mid \mathbf{v}_i = v].$$

We will show by reverse induction on i that $|\alpha_i(v)| \leq \alpha^{k+1-i}$. The claim will then follow by taking i to be 1 and v to be the root of \mathcal{T} . For the base case, we have

$$\alpha_k(v) = \mathbb{E}[h(X_k)b(\mathbf{v}_{k+1}) \mid \mathbf{v}_k = v]$$

$$= \mathbb{E}[h(X_k)b(g_v(X_k))]$$

$$\leq \operatorname{adv}_{h,u}(g_v) \leq \alpha.$$

For the induction step assume that i < k and that $|\alpha_{i+1}(w)| \le \alpha^{k-i}$ for all vertices w in level i+1 of the tree (at distance i from the root). Then, for a vertex v in level i, we have

$$\begin{split} |\alpha_i(v)| &= |\operatorname{E}[h(X_i)h(X_{i+1})\cdots h(X_k)b(\mathbf{v}_{k+1})\mid \mathbf{v}_i = v]| \\ &= |\operatorname{E}[h(X_i)\alpha_{i+1}(g_v(X_i))]| \\ &\leq \operatorname{adv}_{h,\mu}(g_v)\cdot \max_w |\alpha_{i+1}(w)| \\ &< \alpha^{k+1-i}. \end{split}$$

where we used Proposition 34 to justify the second last inequality, and the induction hypothesis to justify the last inequality.

7 Conclusions

In this paper, we presented the first lower bound result for the realistic model of wireless communication networks where there is a restriction on transmission power. Any bit sent by a transmitter is received (with channel noise) only by receivers which are within the transmission radius of the transmitter. We showed that to compute the parity of N input bits with constant probability of error, we need $\Omega(N\log\log N)$ transmissions. This result nicely complements the upper bound result of Ying, Srikant and Dullerud [12], which showed that $O(N\log\log N)$ transmissions are sufficient for computing the sum of all the N bits. Our result also implies that the sum of N bits cannot be approximated up to a constant additive error by any constant error protocol for N(N,R) using $o(N\log\log N)$ transmissions, if $R \le N^{-\beta}$ for some $\beta > 0$.

Although the techniques of network decomposition and translation of bounded protocols to xnd trees are fairly general, some crucial parts of our proof are not. In particular, rearrangement of xnd trees to get ordered xnd trees and analysis of read-once decision trees used the fact that we are trying to compute the parity function. Thus the same proof does not yield similar lower bounds for other functions like majority. In subsequent work, we have eliminated the need for these parts of the proof using entirely different arguments. We have thus succeeded in showing lower bound of $\Omega(N \log \log N)$ transmissions for computing the majority and other functions. These results also show that one cannot approximate the sum of N bits to within an additive error of N^{α} (for some $\alpha > 0$) using $o(N \log \log N)$ transmissions.

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